Critical exponents of the self-avoiding walks on a family of finitely ramified fractals

1987 J. Phys. A: Math. Gen. 20 1215

(http://iopscience.iop.org/0305-4470/20/5/030)

J. Phys. A: Math. Gen. **20** (1987) 1215-1229. Printed in the UK

Critical exponents of the self-avoiding walks on a family of finitely ramified fractals

S Elezović, M Knežević† and S Milošević

Department of Physics and Meteorology, Faculty of Natural and Mathematical Sciences, PO Box 550, 11001 Belgrade, Yugoslavia

Received 27 May 1986, in final form 21 July 1986

Abstract. We have studied the self-avoiding walks (SAW) on a family of finitely ramified fractals. The first member ($b=2$) of the family is the two-dimensional Sierpinski gasket, while the last member ($b=\infty$) appears to be a wedge of the homogeneous triangular lattice. By means of the exact renormalisation group transformations we have calculated the critical exponents α, ν and γ, and the connectivity constant μ, of SAW on each member of a sequence ($2 \leqslant b \leqslant 8$) of the studied fractal family. The obtained exact results are compared with the recent phenomenological proposals and with the results believed to be exact in the case of a homogeneous two-dimensional lattice.

1. Introduction

The self-avoiding walk (SAW) model represents a random walk that must not contain self-intersections. Critical properties of this model on a homogeneous lattice, when the number of steps N tends to infinity, comprise a set of research problems of current interest. There are few well established results. A notable exception appears to be the critical exponent ν for the mean squared end-to-end distance on two-dimensional *homogeneous* lattices. In this case the Flory prediction $\nu=\frac{3}{4}$ (Flory 1953, Fisher 1969) has been analytically (Nienhuis 1982) and numerically (Djordjević *et al* 1983, Rapaport 1985) corroborated. At the same time, SAW on fractals have been studied (see, for example, Rammal *et al* 1984) and two difficult problems have been encountered. One of them has brought about a lot of controversy. It concerns the relation between the universality classes of SAW on disordered (fractal) and SAW on homogeneous (translationally invariant) lattices. The other problem can be formulated as an attempt to establish relations between the critical exponents of SAW on fractals and inherent properties of fractals, such as their fractal ($\bar{d}$) and spectral ($\tilde{d}$) dimensions. This paper is a contribution to understanding the latter problem.

We study SAW on a family of two-dimensional finitely ramified deterministic fractals. The whole family can be generated (Hilfer and Blumen 1984) by an infinite sequence of generators $G(b,d)$, where b is an integer which runs from 2 to infinity and d is a dimension of the Euclidean space in which fractals are embedded (in our case $d=2$). Each $G(b,2)$ is merely an equilateral triangle (see figure 1) that contains b^2 identical smaller triangles of unit side length, out of which only the upward oriented, i.e.

† Temporary address: Ecole Normale Supérieure, Groupe de Physique des Solides, 24 rue Lhomond, 75231 Paris, France.

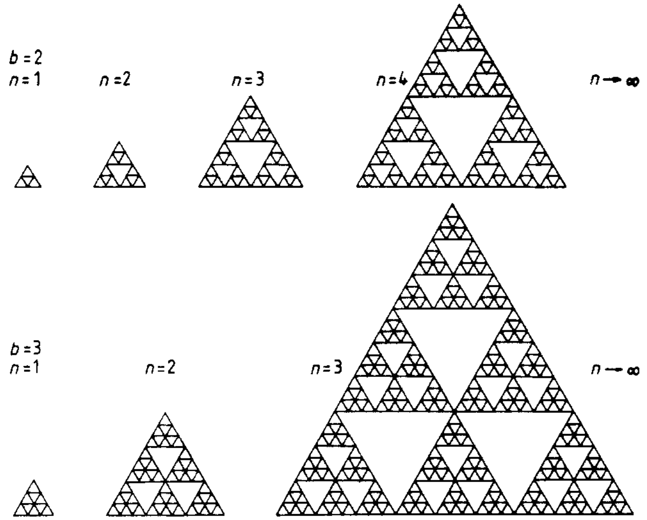

Figure 1. Growing of the fractal lattices. The first row displays the first four stages of growing of the Sierpinski gasket ($b=2$), whereas the second row shows the first three stages of growing of the next member ($b=3$) of the fractal family.

$b(b+1)/2$ altogether, are assumed to be physically present. The fractal related to $G(b,2)$ is obtained in the limit $n\to\infty$ of the process: the structure at stage n results from the stage $(n-1)$ structure after filling with the latter all upward pointing triangles of $G(b,2)$ enlarged by b^n. Thus the first member ($b=2$) of the family is the two-dimensional Sierpinski gasket, whereas we may presume that the subsequent members converge to a wedge of the compact triangular lattice, as the limiting generator $G(\infty,2)$ is nothing other than such a wedge.

The fractal dimensions of the studied fractals are determined (Ben-Avraham and Havlin 1984, Hilfer and Blumen 1984) by the formula

$$\bar{d}=\ln[b(b+1)/2]/\ln b. \tag{1.1}$$

The corresponding spectral dimensions have been tabulated (Hilfer and Blumen 1984) for $2\leqslant b\leqslant 10$ and, in principle, can be calculated for any larger b. Both $\bar{d}$ and $\tilde{d}$ are well defined functions of b and it is our objective to calculate exactly critical exponents of SAW as functions of b as well, assuming that by studying all these functions we will acquire certain knowledge of relations between the exponents and the properties of fractals. To this end in § 2 we shall describe the appropriate renormalisation group (RG) analysis and elaborate on the derivation of the critical exponents α and ν. In § 3 derivation of the critical exponent γ will be worked out. Specific numerical results will be presented in § 4, together with an overall discussion.

2. Critical exponents α and ν

It has been assumed (see, for example, McKenzie 1976) that the total number C_N of distinct SAW of N steps and the total number P_N of distinct SAW loops of N steps, averaged over all possible positions of the starting point, are, for very large N, determined by the power laws

$$C_N \sim \mu^N N^{\gamma-1} \tag{2.1}$$

$$P_N \sim \mu^N N^{\alpha-3} \tag{2.2}$$

where μ is the connectivity constant (the ratio C_{N+1}/C_N approaches μ when $N \to \infty$) and γ and α are the associated critical exponents. By introducing the weight factor x (fugacity) for each step of the walk, the corresponding generating functions are defined to be

$$C(x) = \sum_{N=1}^{\infty} C_N x^N \tag{2.3}$$

$$P(x) = \sum_{N=2}^{\infty} P_N x^N \tag{2.4}$$

so that their leading singular terms, when x approaches $1/\mu$ from below, are of the form

$$C(x) \sim (1 - x\mu)^{-\gamma} \tag{2.5}$$

$$P(x) \sim (1 - x\mu)^{2-\alpha}. \tag{2.6}$$

The above singular behaviour resembles the behaviour of the initial susceptibility and free energy of a magnetic system being close to its critical point. Indeed, it has been shown (de Gennes 1972, Domb 1972, des Cloizeaux 1974) that there is a remarkable correspondence between the SAW model and the n-component spin model in the limit $n \to 0$.

Similarly, the mean squared end-to-end distance $\langle R_N^2 \rangle$ for N-step SAW is expected to obey, for large N, the power law

$$\langle R_N^2 \rangle \sim N^{2\nu} \tag{2.7}$$

so that an appropriate generating function

$$L(x) = \sum_{N=2}^{\infty} \langle R_N^2 \rangle C_N x^N / C(x) \tag{2.8}$$

has the leading singular term

$$L(x) \sim (1 - x\mu)^{-2\nu} \tag{2.9}$$

which parallels the correlation length criticality of the magnetic model system.

In order to determine the critical exponents for SAW on fractals we adopt the RG approach introduced by Dhar (1978) for SAW on the truncated 3-simplex lattice. There are four restricted partition functions (see figure 2) which should be sufficient for

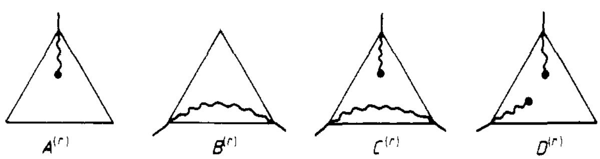

Figure 2. A diagrammatic representation of the restricted generating functions for an rth-order triangle. The interior structure of the triangle is not shown. It is represented by wiggles of the SAW path, which in the $A^{(r)}$ case, for example, starts somewhere within the triangle and leaves it at its upper corner.

representing the generating functions $C(x)$, $P(x)$ and $R(x)$ at an arbitrary (r) stage of the coarse-graining process (figure 3). The starting values (pertinent to a unit triangle) of these functions are

$$A^{(0)}=\sqrt{x} \qquad B^{(0)}=x \qquad C^{(0)}=0 \qquad D^{(0)}=0 \tag{2.10}$$

while their later values $A^{(r)}$, $B^{(r)}$, $C^{(r)}$ and $D^{(r)}$ are sums (weights) of all possible walks within the rth-order triangle consistent with the constraints depicted in figure 2. Since the fractals are self-similar, the recursion relations between two consecutive sets of parameters, i.e. between the $(r+1)$th- and rth-order parameters, should be independent of r. For this reason we can obtain the recursion relations for each member of the studied fractal family by summing all pertinent walks within the corresponding generator. The following relations are evident:

$$B^{(r+1)}=f_b(B^{(r)}) \tag{2.11a}$$

$$A^{(r+1)}=\varphi_b(A^{(r)}, B^{(r)}, C^{(r)}) \tag{2.11b}$$

$$C^{(r+1)}=\psi_b(A^{(r)}, B^{(r)}, C^{(r)}) \tag{2.11c}$$

$$D^{(r+1)}=\phi_b(A^{(r)}, B^{(r)}, C^{(r)}, D^{(r)}). \tag{2.11d}$$

For $b=2$ specific forms of the functions f, φ, ψ and ϕ were found by Dhar (1978). For $b=3$ and $b=4$ we have determined functions f, φ and ψ by a straightforward, but arduous, summing of all possible SAW paths. However, for $b>4$ the work has to be computerised. In appendix 1 we present our results for $3\leqslant b\leqslant 8$, and here we shall study the RG transformations (2.11) for general b.

It follows from (2.11) that the B coordinate of a non-trivial fixed point (A^*, B^*, C^*, D^*) of the RG transformations (2.11) can be found by solving the equation

$$B^*=f_b(B^*) \tag{2.12}$$

independently of the similar RG equations for the other three coordinates. One can verify that for $B^*>0$ equation (2.12) has only one solution which lies in the interval $0<B^*<1$. Indeed, $f_b(B)$ is a sum of the form

$$f_b(B)=\sum_{i=b}^{b(b+1)/2} l_{bi}B^i \tag{2.13}$$

where the coefficients l_{bi} are positive integers (see appendix 1). Hence, the function $y(B)=f_b(B)/B-1$ is a monotonically increasing function from $y(0)=-1$ up to $y(1)>1$ and, on this ground, in the interval $(0, 1)$ there is a unique B^* such that $y(B^*)=0$, i.e.

$$1=\sum_{i=b}^{b(b+1)/2} l_{bi}(B^*)^{i-1}$$

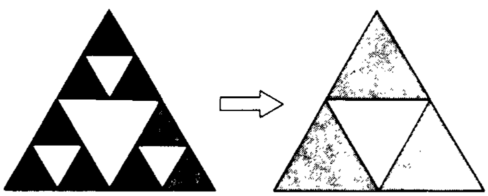

Figure 3. One step of the coarse-graining process in the case of the $b=2$ fractal.

or

$$B^* = \sum_{i=b}^{b(b+1)/2} l_{bi}(B^*)^i \tag{2.14}$$

which proves the uniqueness of the solution of equation (2.12).

Furthermore, by solving equation (2.12) one obtains not only a coordinate of the fixed point but the connectivity constant as well. This can be demonstrated by starting with Dhar's finding (Dhar 1978) for the generating function (2.4) for $b=2$

$$P_2(x) = \sum_{r=1}^{\infty} \frac{1}{3^r}(B^{(r-1)})^3 \tag{2.15}$$

and by verifying

$$P_3(x) = \sum_{r=1}^{\infty} \frac{1}{6^r}[3(B^{(r-1)})^3 + 3(B^{(r-1)})^5 + (B^{(r-1)})^6] \tag{2.16}$$

$$P_4(x) = \sum_{r=1}^{\infty} \frac{1}{10^r}[6(B^{(r-1)})^3 + 9(B^{(r-1)})^5 + 4B^{(r-1)})^6$$
$$+ 12(B^{(r-1)})^7 + 9(B^{(r-1)})^8 + 4(B^{(r-1)})^9 + 3(B^{(r-1)})^{10}] \tag{2.17}$$

for $b=3$ and $b=4$, respectively. Accordingly, for general b one can write

$$P_b(x) = \sum_{r=1}^{\infty} \left(\frac{b(b+1)}{2}\right)^{-r} Y_b(B^{(r-1)}) \tag{2.18}$$

where $Y_b(B^{(r-1)})$ is a polynomial in $B^{(r-1)}$. Equation (2.18) can be rewritten in the form

$$P_b(x) = Y_b(B^{(0)})\left(\frac{b(b+1)}{2}\right)^{-1} + P_b(B^{(1)})\left(\frac{b(b+1)}{2}\right)^{-1} \tag{2.19}$$

which, according to the initial condition (2.10), is equivalent to the transformation

$$P_b(B^{(1)}) = \tfrac{1}{2}b(b+1)P_b(B^{(0)}) - Y_b(B^{(0)}) \tag{2.20}$$

whose fixed points are the fixed points of (2.11*a*). Hence for $B^{(0)}$ less than the non-trivial B^* of (2.12), $B^{(r)}$ decreases and $P_b(x)$ may be finite, whereas for $B^{(0)}$ larger than B^* the successive values of $B^{(r)}$ increases and $P_b(x)$ diverges. For this reason $1/B^*$ can be (Dhar 1978) identified with the connectivity constant μ that appears in equation (2.6).

The critical exponent α, defined by (2.6), can be determined by assuming $x = B^* - \delta$, where δ is a vanishingly small positive number, and by realising that for such an x the polynomial $Y_b(x)$ is finite and $B^{(1)}$ is equal to $x - \lambda_1\delta$, where λ_1 is given by

$$\lambda_1 = \left.\frac{\partial f_b}{\partial B}\right|_{B^*} = \sum_{i=b}^{b(b+1)/2} i l_{bi}(B^*)^{i-1}. \tag{2.21}$$

Inserting these findings into equation (2.20), and keeping in mind (2.6), one obtains

$$\left(\frac{\delta}{B^*}\lambda_1\right)^{2-\alpha} \sim \tfrac{1}{2}b(b+1)\left(\frac{\delta}{B^*}\right)^{2-\alpha} \tag{2.22}$$

or

$$\alpha = 2 - \{\ln[\tfrac{1}{2}b(b+1)]/\ln\lambda_1\}. \tag{2.23}$$

In the next section we present the specific values of λ_1 and α for the studied sequence of fractals.

Knowledge of the eigenvalue λ_1 allows one to calculate the critical exponent ν, providing one assumes that the same exponent ν describes criticality of the end-to-end distance and the correlation length as well. To demonstrate this we apply the scaling argument (see, for example, Stanley *et al* 1982) which asserts that after one step of the renormalisation (see figure 3) all lengths in the rescaled system are reduced by the factor b from the lengths in the original system. Hence the end-to-end distance R transforms as

$$(1-\mu x')^{-\nu} = b^{-1}(1-\mu x)^{-\nu} \tag{2.24}$$

where x and x' are the two successive weights of the SAW step, which means that they can be substituted by, for example, $B^{(0)}$ and $B^{(1)}$, respectively. Thus, from the above equation, and from the established fact $B^* = 1/\mu$, one finds

$$\nu = \ln b/\{\ln[(B^*-B^{(1)})/(B^*-B^{(0)})]\} \tag{2.25}$$

and using the linearised RG transformation (2.11a), i.e. $B^{(1)} - B^* = \lambda_1(B^{(0)} - B^*)$, one finally obtains

$$\nu = \ln b/\ln \lambda_1 . \tag{2.26}$$

The two results, (2.23) and (2.26) for the critical exponents and expression (1.1) for the fractal dimension, can be combined to give

$$\bar{d}\nu = 2-\alpha \tag{2.27}$$

which turns out to be a hyperscaling relation (generalised by the appearance of $\bar{d}$) valid for all fractals of the family studied.

3. Critical exponent γ

Derivation of the critical exponent γ, for an arbitrary member of the fractal family, is a little more complex than the derivation presented of α and ν. First we observe that, in addition to the expression (2.13), the remaining three RG transformations (2.11) should have the following structure:

$$A' = a_1(B)A + a_2(B)C \tag{3.1a}$$

$$C' = c_1(B)A + c_2(B)C \tag{3.1b}$$

$$D' = d_1(B)A^2 + d_2(B)AC + d_3(B)C^2 + d_4(B)D \tag{3.1c}$$

where $a_i(B)$, $c_i(B)$ and $d_i(B)$ are some polynomials in B while $A^{(r+1)}$, $A^{(r)}$, $C^{(r+1)}$, $C^{(r)}$, $D^{(r+1)}$ and $D^{(r)}$ have been abbreviated to A', A, C', C, D' and D, respectively. Next we follow Dhar (1978) and introduce the increment

$$\delta^{(r)} = B^* - B^{(r)} \tag{3.2}$$

which, according to equations (2.11a), (2.13) and (2.21), satisfies, to the lowest order in δ, the recursion relation

$$\delta^{(r+1)} = \lambda_1 \delta^{(r)} . \tag{3.3}$$

The initial value $\delta^{(0)}$ we choose to satisfy the inequality

$$\delta^{(0)} \ll \varepsilon \ll 1 \tag{3.4}$$

where ε is some small positive number. Besides, we define the marker r_0 by the relation

$$r_0 = \ln(\varepsilon/\delta^{(0)})/\ln \lambda_1 \gg 1 \tag{3.5}$$

so that, by comparing the latter with

$$r = \ln(\delta^{(r)}/\delta^{(0)})(\ln \lambda_1)^{-1} \tag{3.6}$$

which follows from (3.3), we can conclude that $\delta^{(r)} < \varepsilon$ for $r < r_0$, and consequently we can replace $B^{(r)}$ in (3.1*a*) and (3.1*b*) by B^*. Hence the transformations become linear:

$$A' = a_1(B^*)A + a_2(B^*)C \tag{3.7}$$

$$C' = c_1(B^*)A + c_2(B^*)C \tag{3.8}$$

and for $1 < r < r_0$ we can write

$$A^{(r)} \simeq K_1 \lambda_2^r \tag{3.9}$$

$$C^{(r)} \simeq K_2 \lambda_2^r \tag{3.10}$$

where K_i $(i = 1, 2, \ldots)$ denote henceforth some constants of proportionality and λ_2 is the larger eigenvalue of the 2×2 matrix formed of the coefficients $a_i(B^*)$ and $c_i(B^*)$, i.e.

$$\lambda_2 = \{a_1(B^*) + c_2(B^*) + [(a_1(B^*) - c_2(B^*))^2 + 4c_1(B^*)a_2(B^*)]^{1/2}\}/2. \tag{3.11}$$

Inserting (3.7), (3.8) and $B = B^*$ in (3.1*c*) we obtain

$$D^{(r+1)} \simeq K_3 \lambda_2^{2r} + d_4(x^*)D^{(r)} \tag{3.12}$$

or

$$D^{(r)} \simeq K_3\{(\lambda_2^2)^r - [d_4(B^*)]^r\}/[\lambda_2^2 - d_4(B^*)] \tag{3.13}$$

which means that the behaviour of $D^{(r)}$ is determined either by λ_2^2 if it is bigger than $d_4(B^*)$, or vice versa. However, one can show that $d_4(B^*)$ is equal to λ_1 given by (2.21). In fact all SAW that contribute to the fourth term on the right-hand side of equation (3.1*c*), i.e. all contributions to $d_4(B)D$, can be obtained from all contributions to $f_b(B)$ by changing each elementary step of type B into an elementary break of type D (see figure 4). Thus each SAW of the length i that contributes to $f_b(B)$ (according to relation (2.13) there are l_{bi} such SAW) changes into i SAW of length $i-1$ with the break D somewhere on the path, so that one can write

$$d_4(B) = \sum_{i=b}^{b(b+1)/2} i l_{bi} B^{i-1} \tag{3.14}$$

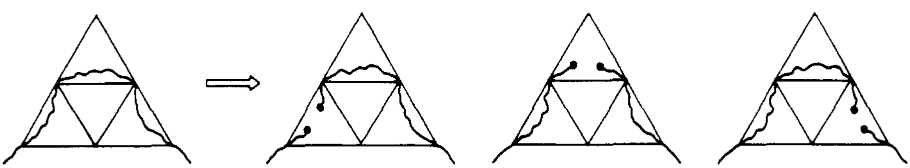

Figure 4. A diagrammatic representation of transforming a $B^{(r)}$ generating function into a group of generating functions of the type $D^{(r)}$.

which proves that $d_4(B^*)$ is equal to λ_1 given by (2.21). Since, on the other hand, we have found that $\lambda_2^2 > \lambda_1$ in each particular case (see table 1), we may finally conclude that $D^{(r)}$ obeys the power law

$$D^{(r)} \simeq K_3' \lambda_2^{2r} \tag{3.15}$$

for $1 < r < r_0$.

For $r > r_0$, $B^{(r)}$ rapidly decreases to zero, and so does $C^{(r)}$ since the polynomials $c_1(B)$ and $c_2(B)$, as can be easily verified, do not contain constant terms. At the same time, the function $A^{(r)}$ rapidly approaches a finite value approximately given by

$$A^{(r)} \simeq K_1(\varepsilon)\lambda_2^{r_0} \tag{3.16}$$

since the constant term of $a_1(B)$ is equal to one and $C^{(r)}$ vanishes. The function $D^{(r)}$ also approaches a finite value

$$D^{(r)} \simeq [A^{(r)}]^2 \tag{3.17}$$

as the constant term of $d_1(B)$ is one, $C^{(r)}$ vanishes and $d_4(B)$ does not have a constant term. Here, $K_1(\varepsilon)$ is a factor of proportionality which depends on the choice of ε but does not depend on δ that measures distance from the fixed value B^*.

The preceding paragraphs provide enough results to calculate the critical exponent γ. The final observation concerns the generating function (2.3). For general b, we argue that $C(x)$ has the following form:

$$C(x) = \sum_{r=1}^{\infty} [b(b+1)/2]^{-r} \{q_1(B^{(r-1)})[A^{(r-1)}]^2 + q_2(B^{(r-1)})A^{(r-1)}C^{(r-1)} + q_3(B^{(r-1)})[C^{(r-1)}]^2 + q_4(B^{(r-1)})D^{(r-1)}\} \tag{3.18}$$

where $q_i(B^{(r-1)})$ are polynomials in $B^{(r-1)}$, such that only $q_1(B^{(r-1)})$ has a constant term, being equal to $3b(b-1)/2$. The above form of $C(x)$ springs from the four possible open paths within the rth-order triangle (see figure 5).

Taking into account results (3.9), (3.10), (3.15), (3.16) and (3.17), and approximating the sum in (3.18) by its largest term we find the asymptotic form

$$C(x) \sim K_4\{\lambda_2^2/[b(b+1)/2]\}^{r_0} \tag{3.19}$$

Table 1. The eigenvalues (2.21) and (3.11) of the RG transformations, and the critical exponents (α, ν, γ) together with the connectivity constant (μ), for SAW on the sequence of the fractals studied. The values for λ_2 and γ for the $b=8$ fractal are missing because generating the necessary RG transformations required a computational time that was beyond our capabilities (see the appendix). The data for the fractal dimension ($\bar{d}$) follow from the formula (1.1), while the spectral dimensions ($\tilde{d}$) are given according to table 1 of Hilfer and Blumen (1984).

b	λ_1	λ_2	α	ν	γ	μ	$\bar{d}$	$\tilde{d}$
2	2.381 97	3.145 60	0.7342	0.7986	1.3752	1.6180	1.5850	1.3652
3	3.991 93	6.639 47	0.7056	0.7936	1.4407	1.8144	1.6309	1.4032
4	5.802 90	11.645 0	0.6905	0.7884	1.4832	1.9750	1.6610	1.4285
5	7.789 84	18.377 9	0.6808	0.7840	1.5171	2.1075	1.6826	1.4471
6	9.936 01	27.058 3	0.6741	0.7803	1.5467	2.2186	1.6992	1.4617
7	12.241 1	37.955 5	0.6697	0.7772	1.5738	2.3128	1.7124	1.4734
8	14.661 2	—	0.6655	0.7744	—	2.3939	1.7233	1.4833

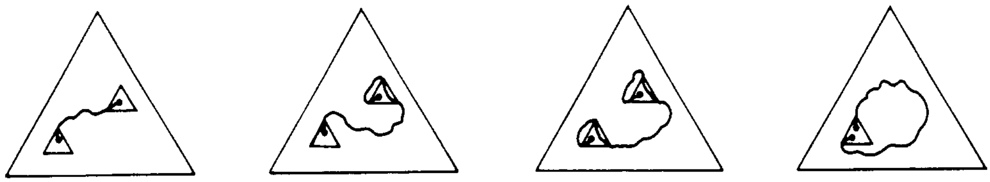

Figure 5. The four possible SAW paths within an rth-order triangle.

which, upon substituting the expression (3.6) for r_0, turns out to be

$$C(x) \sim K_4(\varepsilon/\delta^{(0)})^{\gamma} \tag{3.20}$$

with

$$\gamma = \ln[2\lambda_2^2/b(b+1)]/\ln \lambda_1. \tag{3.21}$$

Therefore, we have succeeded in expressing the three critical exponents α, ν and γ in terms of the two eigenvalues λ_1 and λ_2. In the next section we present particular results for all these quantities for $2 \leq b \leq 7$ (and λ_1, α and ν for $b = 8$) and discuss their significance.

4. Results and discussion

Specific results for the critical exponents α, ν and γ for SAW on the studied fractals, together with the necessary eigenvalues λ_1 and λ_2 and the connectivity constant μ, are given in table 1 (for the sake of completeness we add in table 1 the relevant values of the fractal and spectral dimensions). In view of the fact that the generated RG transformations are exact (see, for example, Dhar 1978, Rammal *et al* 1984), the presented results should be regarded as exact as well. Thus, the best way to assess a possible significance of these results is to make comparisons with existing phenomenological predictions.

We start our discussion with the critical exponent ν, as it has been been most frequently studied in the past. In figure 6(a) we have depicted the exact results for ν and the following two phenomenological proposals:

$$\nu_{\mathrm{F}} = 3/(2+\bar{d}) \tag{4.1}$$

and

$$\nu_{\mathrm{RTV}} = 3\tilde{d}/\bar{d}(2+\tilde{d}). \tag{4.2}$$

The former was proposed by Kremer (1981) and Sahimi (1984) and appears to be a straightforward generalisation of the self-consistent Flory formula $\nu = 3/(2+d)$ (Flory 1953, Fisher 1969). The formula (4.2) was surmised by Rammal *et al* (1984) and almost simultaneously by Sahimi (1984). There is an additional closed-form expression for ν:

$$\nu_{\mathrm{A}} = (4+\bar{d})/4\bar{d} \tag{4.3}$$

proposed by Alexandrowicz (1984), who assumed (4.3) to be valid in the region $4/3 \leq \bar{d} \leq 4$, which certainly includes fractal dimensions of all members of the studied fractal family. However, values of ν that follow from (4.3) lie well above those obtained from (4.1) (for $2 \leq b \leq 4$ the former lie out of the frame accepted in figure 6(a)), and for this reason ν_{A} is not plotted in figure 6(a). On the whole, it is evident that none of the quoted formulae matches the exact results for ν (except for $b = 2$, ν_{F} is a better

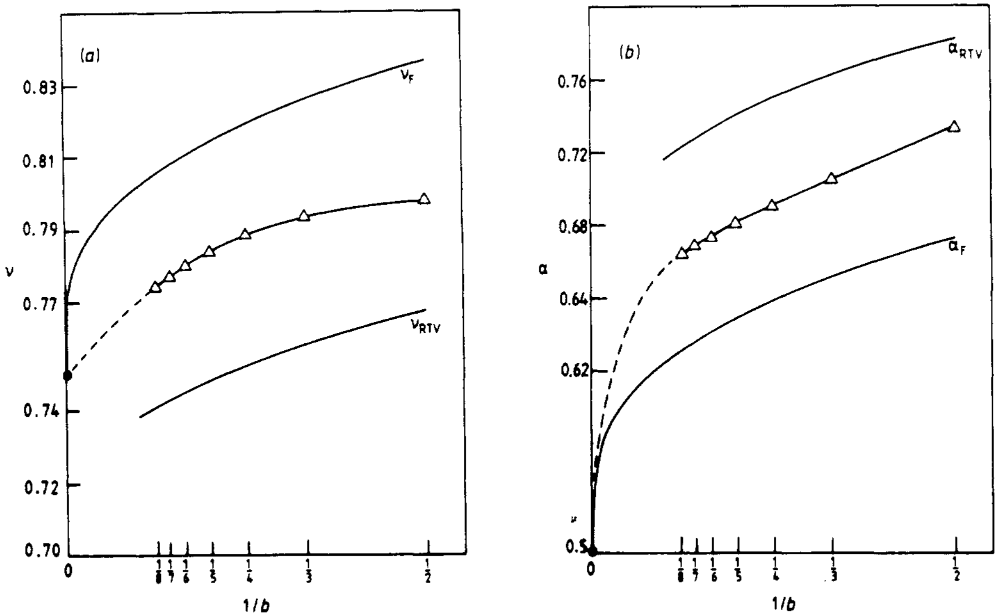

Figure 6. (a) The critical exponent ν as a function of the parameter b which enumerates all members of the studied family of fractals. The full curve which connects the obtained exact results (△) serves merely as a guide to the eye, as well as its broken continuation. The curves labelled by ν_F and ν_{RTV} represents the phenomenological proposals (4.1) and (4.2), respectively. The curve ν_{RTV} is drawn in the interval of b ($2 \leqslant b \leqslant 10$) for which the spectral dimension $\tilde{d}$ has been tabulated (Hilfer and Blumen 1984). The full circle corresponds to the result believed to be exact in the case of a two-dimensional homogeneous lattice (Nienhuis 1982). (b) The exact results and phenomenological predictions for the critical exponent α. The labelling is the same as in the case of ν.

approximation than ν_{RTV}, although it was conjectured (Rammal *et al* 1984) that ν_{RTV} might be an improvement on ν_F). Therefore, one may observe that the critical exponent ν for SAW on fractals can hardly be a simple (rational) function of the fractal ($\bar{d}$) and spectral dimensions ($\tilde{d}$) alone and, on the strength of this observation, one may conclude that additional intrinsic properties of fractals should be taken into account in further attempts to establish a general formula for ν.

Before passing on to the critical exponents α and γ we note that all three expressions (4.1), (4.2) and (4.3), when $\bar{d} \to 2$ (i.e. when $b \to \infty$ in the case under study), converge to $\frac{3}{4}$, which is the value of ν believed to be exact for two-dimensional homogeneous lattices (Nienhuis 1982). As regards the exact values of ν for the studied fractals, we may expect that they converge to $\frac{3}{4}$ (when $b \to \infty$) as well, since the last member of the fractal family appears to be a wedge of a homogeneous triangular lattice and it has been corroborated (Guttmann and Torrie 1984, Cardy and Redner 1984, De'Bell and Lookman 1985) that the critical exponents of a wedge and the bulk should be the same. Moreover, a comparison of our figure 6(a) with figure 2(a) of Djordjević *et al* (1983) reveals that convergence of *exact* values of ν, obtained for the fractals, to $\frac{3}{4}$ is at least as convincing as the convergence of *effective* values of ν, obtained by the series analysis methods for the homogeneous triangular lattice.

In figure 6(b) we delineate exact results for the critical exponent α, obtained according to formula (2.23), and two phenomenological predictions that follow from formulae (4.1) and (4.2), under the assumption that the latter are compatible with the relation (2.27). Again, it is evident that the phenomenological predictions do not tally with the presented exact results, although a better agreement could be expected for large b, i.e. when $\bar{d}$ approaches 2, in particular between exact results and the Flory type prediction α_F. In connection with this expectation, it can be seen that the obtained exact results for α have not invaded the asymptotic region close to $\alpha = \frac{1}{2}$ which is assumed to be exact for $d = 2$. Apparently, additional calculations (for $b > 8$) are necessary in order to demonstrate that the exact results for α converge, for large b, to $\frac{1}{2}$.

The above conclusion is even more pertinent to the case of the critical exponent γ which is depicted in figure 7(a). This is not because of the completely different behaviour of the exact results and the phenomenological proposals, which will be commented on later, but because it seems that the exact results for fractals can never approach the value $\gamma = \frac{43}{32}$ assumed to be exact for the two-dimensional translationally invariant lattices (Nienhuis 1982). Likewise, it may seem that the total number of SAW, given by expression (2.1), could become larger for fractals (because of larger critical exponents γ) than C_N for the homogeneous triangular lattice. However, such alarm is not tenable, even if γ continues to increase (for $b > 7$), since the behaviour of C_N is governed by the term μ^N and from figure 7(b) it is plausible that the connectivity constant μ for the fractals is never larger than $\mu \simeq 4.149$ for the triangular lattice (De'Bell and Lookman 1985). On the other hand, we may expect that γ for the fractals will start to decrease after reaching a maximum for $b > 7$. This expectation springs from the established relation of the SAW model with the $n \to 0$ limit of the n-vector

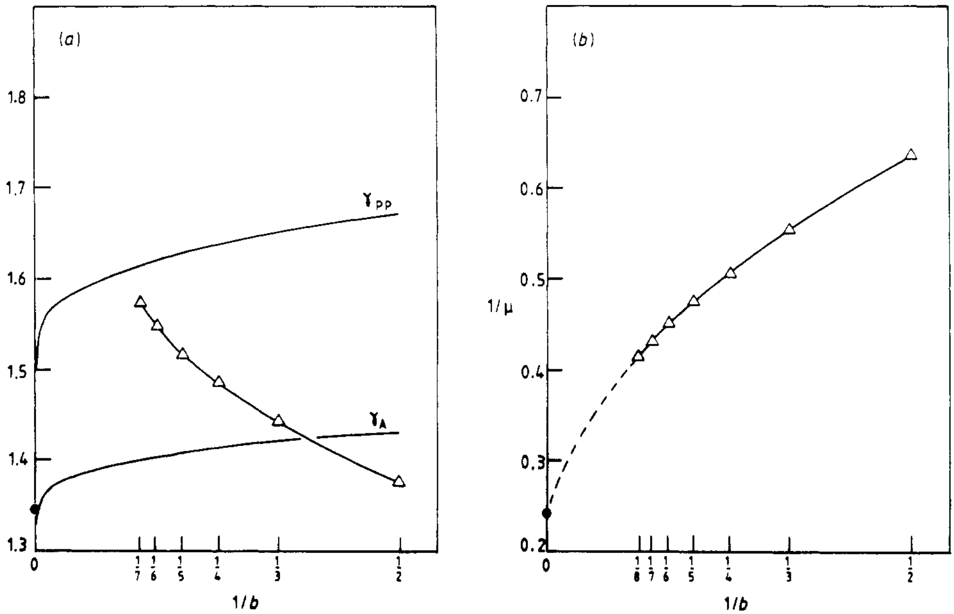

Figure 7. (a) The critical exponent γ as a function of b. It is expected that the exact results (△) will reach a maximum for a $b > 7$ and start to converge to the result (●) presumed to be exact for a two-dimensional homogeneous lattice (see the text). The curves labelled by γ_A and γ_{PP} represent phenomenological proposals (4.4) and (4.5), respectively. (b) The reciprocal of the connectivity constant μ. The labelling is the same as in the case of γ.

model and from the general behaviour of the critical exponent γ as a function of n and the *continuous* dimension d (see, for example, figure 3 of Fisher (1974)). There are valleys in the corresponding contours of γ which are convex towards the d axis and thus strongly suggest that for the fixed n $(n=0)$ one may expect a maximum of γ for a certain fractional dimension. Consequently, we anticipate that additional calculations will show that the exact values of γ for the fractals will start to approach, for large b, the exact value $(\frac{43}{32})$ for the triangular lattice or, possibly, a lower value that is pertinent to a section of the triangular lattice (De'Bell and Lookman 1985).

Concerning the comparison of the exact results for γ with the phenomenological proposals, we observe that neither the formula

$$\gamma_{\mathrm{A}}=8/(4+\bar{d}) \tag{4.4}$$

nor the formula

$$\gamma_{\mathrm{PP}}=6/(2+\bar{d}) \tag{4.5}$$

predicts the behaviour displayed by the exact results. The former was derived (Alexandrowicz 1984) from random walks through a progressive exclusion of walks with loops (and is connected with the formula (4.3) by the relation $\gamma\nu\bar{d}=2$), whereas the latter appears to be a generalisation of the recent completion (Pietronero and Peliti 1985) of the Flory self-consistent approach. One should also observe that both formulae (4.4) and (4.5) do not predict the presumably exact value $\gamma=\frac{43}{32}$ for the triangular lattice, although γ_{A} provides a better approximation. Besides, neither of the formulae agrees with the shape pertinent to the Fisher contours (Fisher 1974) for γ, in the (d, n) plane, in the entire interval $1\leqslant d\leqslant 4$. Therefore, if the formulae γ_{A} and γ_{PP} provide reasonable approximations for the critical exponent γ of SAW on fractals, it can only be for fractals embedded in the Euclidean spaces with $d>2$ and, thereby, not for the fractals studied in this paper.

In conclusion, we point out that in this paper we have obtained exact results for the critical exponents of SAW on a sequence of an infinite class of two-dimensional fractals. These results show that the critical exponents cannot be represented by simple functions of the fractal and spectral dimensions alone. In addition, we argue that the obtained results, particularly those for the critical exponent ν, demonstrate that by studying SAW on fractals, i.e. on dilatationally invariant lattice, one can probe via exact results the criticality of SAW on a homogeneous (translationally invariant) lattice.

Acknowledgments

We would like to thank Mr Miloš Vićić for cordial assistance in the computational part of this work. One of us (MK) is grateful to Professor J Vannimenus for helpful discussions.

This work has been supported in part through funds provided by the Yugoslav-USA Joint Scientific Board under project no 534 (NSF) and by the Serbian Science Foundation under project no 1.2.

Appendix

Here we present coefficients of the RG transformations (2.11a), (3.1a) and (3.1b). For the sake of compactness, we give the following closed-form expressions:

$$B' = \sum_{i=b}^{b(b+1)/2} l_{bi} B^i \tag{A1}$$

$$a_1(B) = \sum_{i=0}^{b(b+1)/2-1} p_{bi} B^i$$

$$a_2(B) = \sum_{i=2}^{b(b+1)/2-1} q_{bi} B^i \tag{A2}$$

$$c_1(B) = \sum_{i=b}^{b(b+1)/2-1} s_{bi} B^i$$

$$c_2(B) = \sum_{i=m}^{b(b+1)/2-1} t_{bi} B^i \tag{A3}$$

where in the last sum m is equal to 2 and 4 for $b=2$ and $b=3$, respectively, whereas $m=b+2$ for $4 \leqslant b \leqslant 7$.

The coefficients l_{bi}, p_{bi}, q_{bi}, s_{bi} and t_{bi} can be calculated straightforwardly for $2 \leqslant b \leqslant 4$, whereas for $b>4$ the calculation has to be computerised. Thus, the $b=5$ and $b=6$ coefficients can be calculated on a personal computer (for example, on the ZX Spectrum computer). On the other hand, calculation of the sets $\{l_{7i}\}$ and $\{p_{7i}, q_{7i}, s_{7i}, t_{7i}\}$ required 0.5 and 3 h on the PDP 11/70 computer respectively, while calculation of the set $\{l_{8i}\}$ required 24 h on the same computer.

$\{l_{2i}\} = \{1, 1\}$

$\{l_{3i}\} = \{1, 3, 1, 2\}$

$\{l_{4i}\} = \{1, 6, 6, 9, 9, 9, 4\}$

$\{l_{5i}\} = \{1, 10, 20, 30, 54, 68, 98, 94, 86, 38, 16\}$

$\{l_{6i}\} = \{1, 15, 50, 90, 201, 327, 604, 898, 1392, 1687, 1985, 1720, 1371, 757, 334, 68\}$

$\{l_{7i}\} = \{1, 21, 105, 245, 605, 1244, 2555, 4850, 9028, 15\,551, 25\,621, 38\,111, 52\,865, 64\,698, 71\,615, 69\,027, 58\,384, 41\,413, 23\,883, 10\,380, 3038, 464\}$

$\{l_{8i}\} = \{1, 28, 196, 602, 1625, 4012, 9044, 20\,061, 41\,929, 86\,166, 168\,393, 319\,136, 571\,346, 974\,364, 1553\,648, 2326\,497, 3220\,025, 4140\,261, 4871\,926, 5259\,672, 5112\,431, 4457\,613, 3397\,446, 2235\,226, 1222\,228, 538\,776, 176\,598, 38\,418, 3838\}$

$\{p_{2i}\} = \{1, 2, 2\}$

$\{p_{3i}\} = \{1, 2, 6, 8, 8, 6\}$

$\{p_{4i}\} = \{1, 2, 6, 16, 32, 46, 62, 64, 50, 24\}$

$\{p_{5i}\} = \{1, 2, 6, 16, 48, 110, 206, 358, 558, 774, 914, 902, 714, 380, 134\}$

$\{p_{6i}\} = \{1, 2, 6, 16, 48, 142, 366, 806, 1652, 3150, 5622, 9302, 14\,000, 19\,140, 23\,208, 24\,808, 22\,320, 16\,756, 9732, 4102, 932\}$

$\{p_{7i}\} = \{1, 2, 6, 16, 48, 142, 430, 1190, 2932, 6740, 14\,660, 30\,372, 59\,956, 112\,296, 198\,214, 328\,106, 506\,598, 724\,656, 951\,780, 1136\,934, 1220\,670, 1163\,212, 966\,802, 682\,910, 396\,204, 176\,160, 54\,190, 8656\}$

$\{q_{2i}\} = \{2\}$

$\{q_{3i}\} = \{2, 4, 4, 10\}$

$\{q_{4i}\} = \{2, 4, 12, 22, 46, 60, 54, 42\}$

$\{q_{5i}\} = \{2, 4, 12, 38, 82, 188, 358, 590, 820, 894, 866, 492, 228\}$

$\{q_{6i}\} = \{2, 4, 12, 38, 114, 288, 706, 1610, 3228, 6234, 10\,290, 15\,810, 20\,878, 24\,814, 24\,172, 19\,912, 12\,508, 5810, 1460\}$

$\{q_{7i}\} = \{2, 4, 12, 38, 114, 352, 966, 2546, 6328, 14\,622, 32\,206, 66\,448, 128\,702, 230\,660, 385\,328, 592\,798, 836\,572, 1067\,984, 1223\,338, 1236\,980, 1092\,246, 814\,010, 500\,688, 233\,778, 75\,556, 12\,524\}$

$\{s_{2i}\} = \{1\}$

$\{s_{3i}\} = \{1, 5, 4\}$

$\{s_{4i}\} = \{1, 8, 24, 44, 35, 20\}$

$\{s_{5i}\} = \{1, 12, 46, 146, 343, 544, 637, 568, 366, 116\}$

$\{s_{6i}\} = \{1, 17, 86, 296, 969, 2580, 5604, 9658, 14\,224, 17\,706, 18\,569, 15\,569, 9875, 4034, 886\}$

$\{s_{7i}\} = \{1, 23, 153, 597, 2109, 6754, 19\,770, 50\,126, 109\,890, 211\,291, 363\,649, 561\,131, 776\,409, 952\,812, 1021\,693, 934\,685, 707\,334, 425\,579, 192\,674, 58\,016, 8372\}$

$\{t_{2i}\} = \{3\}$

$\{t_{3i}\} = \{9, 8\}$

$\{t_{4i}\} = \{31, 64, 43, 40\}$

$\{t_{5i}\} = \{2, 123, 374, 565, 780, 772, 552, 196\}$

$\{t_{6i}\} = \{2, 34, 523, 2074, 4808, 9060, 14\,665, 19\,718, 22\,663, 20\,368, 13\,597, 5778, 1408\}$

$\{t_{7i}\} = \{2, 46, 306, 2475, 11\,622, 34\,560, 83\,292, 175\,296, 323\,896, 538\,266, 795\,044, 1039\,666, 1179\,286, 1132\,484, 892\,030, 558\,073, 261\,328, 80\,910, 11\,804\}.$

Note added in proof. We have obtained λ_2 and γ for the $b = 8$ fractal. The corresponding results are $\lambda_2 = 51.3531$ and $\gamma = 1.5991$. We are grateful to Professor H E Stanley (Boston University) who provided an opportunity for one of the authors (SM) to perform the relevant calculation at the Boston University Academic Computing Center.

References

Alexandrowicz Z 1984 *Phys. Rev. Lett.* **53** 1088
Ben-Avraham D and Havlin S 1984 *Phys. Rev.* A **29** 2309
Cardy J L and Redner S 1984 *J. Phys. A: Math. Gen.* **17** L933
De'Bell K and Lookman T 1985 *J. Phys. A: Math. Gen.* **18** L457
des Cloizeaux J 1974 *Phys. Rev.* A **10** 1665
Dhar D 1978 *J. Math. Phys.* **19** 5
Djordjević Z V, Majid I, Stanley H E and dos Santos R J 1983 *J. Phys. A: Math. Gen.* **16** L519
Domb C 1972 *J. Phys. C: Solid State Phys.* **5** 1399

Fisher M E 1969 *J. Phys. Soc. Japan (Suppl.)* **26** 44
—— 1974 *Rev. Mod. Phys.* **46** 597
Flory P J 1953 *Principles of Polymer Chemistry* (Ithaca, NY: Cornell University Press)
de Gennes P G 1972 *Phys. Lett.* **38A** 339
Guttmann A J and Torrie G M 1984 *J. Phys. A: Math. Gen.* **17** 3539
Hilfer R and Blumen A 1984 *J. Phys. A: Math. Gen.* **17** L537
Kremer K 1981 *Z. Phys.* B **45** 149
McKenzie D S 1976 *Phys. Rep.* **27C** 35
Nienhuis B 1982 *Phys. Rev. Lett.* **49** 1062
Pietronero L and Peliti L 1985 *Phys. Rev. Lett.* **55** 1479
Rammal R, Toulouse G and Vannimenus J 1984 *J. Physqiue* **45** 389
Rapaport D C 1985 *J. Phys. A: Math. Gen.* **18** L201
Sahimi M 1984 *J. Phys. A: Math. Gen.* **17** L379
Stanley H E, Reynolds P J, Redner S and Family F 1982 *Real Space Renormalization* ed T W Burkhardt and J M J van Leeuwen (Berlin: Springer) pp 169-206